BELLCOMM, INC.

SUBJECT: An Explanation of the Secant Method as Proposed by Wolfe for Solving a System of Simultaneous Nonlinear Equations - Case 103-4

DATE: March 1, 1966

FROM: J. J. Schoch

ABSTRACT

The secant method as proposed by P. Wolfe (Ref. 1) is explained in such a manner as to make it intelligible for those users of the method not having a too profound mathematical background. The method is available in the Bellcomm computer library as Subroutine Wolfe and is presently used in the trajectory program described in Reference 2.

The method is explained first for one then for two dimensions and the relationships are finally extended to n dimensions. It is hoped that by understanding how the method works the reader may make a more intelligent use of it.

[redacted]

(NASA-CR-156199) AN EXPLANATION OF THE SECANT METHOD AS PROPOSED BY WOLFE FOR SOLVING A SYSTEM OF SIMULTANECUS NONLINEAR EQUATIONS (Bellcomm, Inc.) 18 p

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MEMORANDUM FOR FILE

Introduction

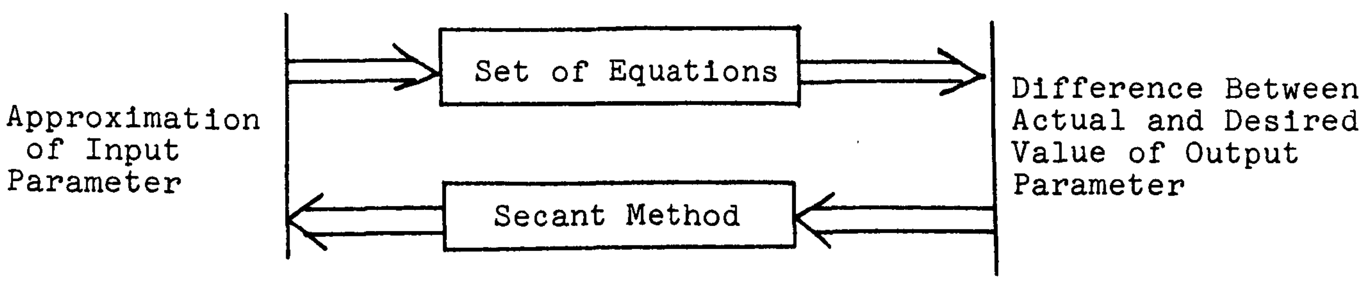
The secant method for solution of simultaneous nonlinear equations is explained very briefly in Reference 1. Unfortunately, this reference does not, in its scant 1-1/2 page article, show the geometric significance of the method, nor does it, in its very succinct treatment of the subject, give many explanations. The reader with limited mathematical background will therefore not get too much out of it.

This memorandum was written for the unique purpose of more fully explaining the method. It does not propose any improvements.

Purpose and Use of the Secant Method

The secant method is useful whenever it is desired to compute some quantities by trial and error. Typically, it may be desired to determine by trial and error some input parameters to a function in such a manner that certain output parameters reach a desired value.

Schematically, this may be represented as shown below:



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The block representing the "Set of Equations" may be thought of as a device which, for a given approximation of the input parameters, computes the difference between the actual and desired value of the output parameters. The block representing the "Secant Method", on the other hand, represents a device which, for a given set of errors of output parameters, provides a better approximation of the input parameters. It has to be assumed, of course, that the equations cannot be solved in reverse, i.e., the input parameters cannot be derived directly by entering the desired output parameters in the equations because in this case there would not be a purpose in using the method. The method will be explained by using two examples. The first example applies the method to a one-dimensional problem, the second to a problem in two dimensions. The examples selected are trivial. Their sole purpose is to explain the method.

One-Dimensional Example

Find the number, x, whose square is 30. (It has been assumed that the problem may not be solved by determining the square root of 30.)

Let the first approximation of the number whose square is 30 be 4. Refer now to Figure 1 on which the difference between the actual and desired value, i.e., the error $E_o = x^2 - 30$ is plotted versus x. It will now be shown how, by using the secant method, a better guess for x may be determined.

The first guess, x = 4, is represented by point P_1. It provides $x^2 = 16$ and the error E_{o1} represented by the segment P_1A_1 is $E_{o1} = 16\text{-}30 = -14$. An additional point in the neighborhood of the first one is selected. Let this be point P_2 having x = 2. Its error, E_{o2}, represented by the segment P_2A_2, is $E_{o2} = 4\text{-}30 = -26$. The straight line through points A_1 and A_2 is the secant to the parabola $y = x^2$. It intersects the $E_o = 0$ line at a point P_3 corresponding to an $x = 6.3333$. This new value is a little closer than the original guess, point P_1. It is used as the second approximation. Corresponding error, E_{o3}, represented by segment P_3A_3, is $E_{o3} = 40.11-30 = 10.11$. One of the two old points P_1 and P_2 having the largest error is now discarded and a new secant drawn connecting the remaining one, A_1, with the new approximation A_3. This new secant intersects the $E_o = 0$ line at point P_4 having an x value of 5.35. This point P_4 is the third approximation. The procedure could be continued by discarding each time the one of the two old points having the largest error and drawing the new secant by connecting the remaining one with the latest approximation until the desired accuracy is obtained.

Note that the first time that the secant method is used it is necessary to pick a second point. For all other iterations, the point representing the last guess is connected to the best of the previous ones.

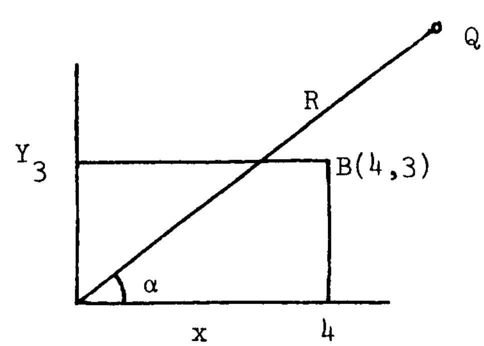
For this one-dimensional case the procedure is simple and straightforward. In the next subsections, it will be shown how the same principle is applicable to two or more dimensions.

Two-Dimensional Example

It is desired to find a point B of given Cartesian coordinates (4,3) by defining the radius R and the anomaly α. Given R and α the Cartesian coordinates are provided by

$$x = R \cos \alpha$$
$$y = R \sin \alpha$$

or the errors E_1 and E_2 between actual and desired value of x and y by:



$$E_1 = x - 4 = (R \cos \alpha) - 4$$
$$E_2 = y - 3 = (R \sin \alpha) - 3 \qquad (1)$$

The problem consists in making E_1 and E_2 the errors in the x and y direction as small as possible.

It will, of course, be assumed that it is not possible to explicitly solve the Equations (1) above for R and α.

Given an initial approximation for R and α it will be shown how the secant method determines a better one. The method is essentially the same as before except that due to the added dimension secant planes will have to be drawn instead of secant lines.

Let the first guess be:

$$R = 10$$
$$\alpha = 30^\circ \qquad (2)$$

The value of x obtained from the first equation of (1) by substituting the values given in (2) is 8.66 and the error $E_{11} = 4.66$.

Reference is now made to Figure 2. It is an isometric sketch showing E_1, the error in x, as a function of the input parameters R and α. For the given initial guess represented by the point Q_1 (R = 10, α = 30) the error in x E_{11} is given by the vertical distance $\overline{Q_1 P_1}$. A second point Q_2 will be selected in the R, α plane which has a slightly different R, say R = 10.625, but the same α. Let Q_2 be that point. The error in x, which may be called E_{12}, is in this case 5.201 and is represented by the vertical distance $\overline{P_2 Q_2}$. Due to the added dimension, a third point Q_3 is now selected which has the same R value as Q_2 but a slightly different value of α, say 31.9. This point Q_3 provides an error E_{13} of 5.022 represented by the segment $\overline{Q_3 P_3}$. The points P_1, P_2, and P_3 are three points of the surface representing E_1, the error in x. The plane through the points P_1, P_2, and P_3 is a secant plane to the E_1 surface. By extending the P_1P_2 line in the $\alpha = 30^\circ$ plane (parallel to the E_1, R axes), one obtains point C_1 as an intersection of that line with the R,α

plane. Similarly, point D_1 is obtained by extending the P_2P_3 line in the R = 10.625 plane (parallel to the E_1, α axes) to intersect R,α plane. A line a_1 connecting points C_1 and D_1 is the intersection between the secant plane to the surface E_1 and the R,α plane. Any point of line a_1 would be expected to give a smaller error for E_1 than any of the initially selected points Q_1, Q_2, and Q_3. As will be explained later, this is not always necessarily so.

The surface E_1 represents the error in x. Refer now to Figure 3. If the surface E_2 represents the error y, then for the same points Q_1, Q_2, and Q_3 errors E_{21}, E_{22}, and E_{23} are represented by the segments $\overline{Q_1S_1}$, $\overline{Q_2S_2}$, $\overline{Q_3S_3}$ in an R,α,E_2 space.

As in the R, α,E_1 space a secant plane to the E_2 surface may be laid through the points S_1, S_2, and S_3. This plane is found to intersect the R,α plane along a line a_2 going through points C_2 and D_2. Any point along line a_2 used for computing Equations (1) should give a smaller error in E_2 than the ones obtained using points Q_1, Q_2, Q_3.

On Figure 2 line a_2 from Figure 3 has been drawn. Since any point on line a_1 indicates a better point for surface E_1 and any point on a_2 a better point for surface E_2, then the intersection of the two lines, point Q_4, should provide smaller errors for both E_1 and E_2. Point Q_4 is taken as the new approximation. In the present calculation, point Q_4 has the coordinates $R = 4.974$; $\alpha = 32.3°$. For these values, Equations (1) provide an x and y value from which $E_{14} = x - 4 = .1607$ and $E_{24} = y - 3 = -.2746$. These errors are indeed considerably smaller than any of those obtained using points Q_1, Q_2, Q_3.

At this point in the calculation a decision has to be taken as to which of the old points should be eliminated. In the one-dimensional case it is simple to designate the poorest estimate. In the case of two or more dimensions it is a little more complicated. In Reference 1 the point having the largest sum of the squares of the errors is eliminated. In the Bellcomm version of this subroutine (Reference 3) a more complex criterion is used. With the nomenclature used above where the errors in x are designated E_{1i} and the ones in y designated E_{2i} (with i designating the point) the quantities

$$K_1 = \frac{E_{11}^2}{E_{11}^2 + E_{12}^2 + E_{13}^2} + \frac{E_{21}^2}{E_{21}^2 + E_{22}^2 + E_{23}^2}$$

$$K_2 = \frac{E_{12}^2}{E_{11}^2 + E_{12}^2 + E_{13}^2} + \frac{E_{22}^2}{E_{21}^2 + E_{22}^2 + E_{23}^2} \quad (3)$$

$$K_3 = \frac{E_{13}^2}{E_{11}^2 + E_{12}^2 + E_{13}^2} + \frac{E_{23}^2}{E_{21}^2 + E_{22}^2 + E_{23}^2}$$

are computed and the point having the largest K eliminated. This method adds for each point Q_1, Q_2, and Q_3 the relative errors in the two spaces and compares their sums.

Figure 4 shows the relative position of the points Q_i in the R,α plane. The three initial points Q_1, Q_2, and Q_3 were used to provide a new approximation: point Q_4. Then, through the use of Equation 3 point Q_3 is eliminated and the calculation is repeated using points Q_1, Q_2, and Q_4. A secant plane may be laid through points P_1, P_2, and P_4 in the R,α,E_1 space and another through points S_1, S_2, and S_4 in the R,α,E_2 space. The intersection of these two secant planes with the R,α plane provides point Q_5. Through the use of Equations 3 point Q_2 is eliminated. The procedure is repeated using points

Q_1, Q_4, and Q_5 with point Q_6 determined. The procedure may be continued until the errors E_1 and E_2 are both smaller than some desired value.

At this point it may be interesting to note the difference between the one and the two-dimensional example. In the one-dimensional case, the new approximation is found as an intersection between a secant line and the zero error line. In the two-dimensional example, it is found by intersecting the secant planes defined by the errors of the two equations with the zero error plane.

In the description given so far, it has always been implied that every new guess provides a smaller error than any of the previous ones. Although this was also the case for the two given examples, it will not always be so, nor is this necessary for convergence. Taking the two-dimensional example it will occasionally happen that due to the relative position of the secant plane, with respect to the error surface E_1 or E_2, the new guess will provide a larger error than the previous one. In such a case, however, an improvement is expected in the next iteration. After the "worst" of the old points has been discarded and the remaining ones combined with that new guess the new intersection of the two new secant planes with the R,α plane may very well provide a much better point and make the process converge.

In case the initial guess is too far from the final solution, the method will not converge. There is no quantitative criterion to establish how close the initial guess has to be from the final solution; however, in the two-dimensional case an initial error of 100% is acceptable most of the time.

Generalization to an n-Dimensional Example

In the previous subsection the secant method was explained as applied to a two-dimensional problem by using purely geometric concepts. Using these geometric concepts formulae will be derived. These formulae will be later extended to n dimensions.

In the previous subsection the three points Q_1, Q_2, and Q_3 provided, by solving Equations 1, the points

$$P_1(R_1, \alpha_1, E_{11})$$

$$P_2(R_2, \alpha_2, E_{12})$$

$$P_3(R_3, \alpha_3, E_{13})$$

in the R, α, E_1 space and the points

$$S_1(R_1, \alpha_1, E_{21})$$

$$S_2(R_2, \alpha_2, E_{22})$$

$$S_3(R_3, \alpha_3, E_{23})$$

in the R, α, E_2 space.

For a new guess, say point Q_4 $(R_4, \alpha_4, 0)$ in the R, α plane, to belong to the plane through points P_1, P_2, P_3 it is necessary and sufficient* that its coordinates satisfy the conditions:

$$R_4 = \pi_1 R_1 + \pi_2 R_2 + \pi_3 R_3 \tag{4}$$

$$\alpha_4 = \pi_1 \alpha_1 + \pi_2 \alpha_2 + \pi_3 \alpha_3 \tag{5}$$

$$0 = \pi_1 E_{11} + \pi_2 E_{12} + \pi_3 E_{13} \tag{6}$$

---

*See, for instance, page 50 of Reference 4.

where $$\pi_1 + \pi_2 + \pi_3 = 1 \tag{7}$$

Furthermore, for that same point to belong also to the plane through points S_1, S_2, S_3 it is necessary and sufficient that its coordinates satisfy the conditions

$$R_4 = \pi_1 R_1 + \pi_2 R_2 + \pi_3 R_3 \tag{8}$$

$$\alpha_4 = \pi_1 \alpha_1 + \pi_2 \alpha_2 + \pi_3 \alpha_3 \tag{9}$$

$$0 = \pi_1 E_{21} + \pi_2 E_{22} + \pi_3 E_{23} \tag{10}$$

$$\pi_1 + \pi_2 + \pi_3 = 1 \tag{11}$$

It will be noted that Equations 4, 5, and 7 are identical to Equations 8, 9, and 11. Consequently, Equations 6, 10, and 11 may be used to compute the parameters π_1, π_2, and π_3 and Equations 4 and 5 to obtain the values of the coordinates of the new guess, i.e., R_4 and α_4. It has now been shown how the laborious process of fitting secant planes through the error surfaces E_1 and E_2 have been replaced by solving a system of linear equations (Equations 6, 10, and 11) in π_1, π_2, and π_3 and substituting these values of π in another set of linear equations to obtain the coordinates of the new guess.

In going to a problem having more than two dimensions, the geometric approach becomes very difficult to visualize. It is, however, very easy to extend the analytical method to a problem in n dimensions.

Let it be assumed that it is desired to determine n functions

$$f_i(x) \qquad i = 1, \ldots , n,$$

where x represents n independent variables $x_1, x_2, x_3, \ldots, x_n$. The value of the functions f_i should approach a certain value or the error between the desired and actual value of $f_i(x)$ should be small.

Let E_i be the error where i may go from 1 to n. The initial guess may be represented by a point x_{i1} and it is necessary to select n more initial points by perturbing each of the n dimensions, therefore obtaining n + 1 points x_{ij} where $j = 1, \ldots, n + 1$.

Geometrically, the intersection of the n secant hyperplanes with the zero error hyperplane will give the n coordinates of the new guess.

Using the analytical method the n functions $f_i(x)$ are solved for n + 1 points providing an array of n x (n+1) errors $E_{i,j}$ where the index i identifies the equation and the index j the point.

The linear system of n + 1 equations

$$
\begin{aligned}
\pi_1 E_{1,1} + \pi_2 E_{1,2} + \cdots + \pi_{n+1} E_{1,n+1} &= 0 \\
\pi_1 E_{2,1} + \pi_2 E_{2,2} + \cdots + \pi_{n+1} E_{2,n+1} &= 0 \\
&\vdots \\
\pi_1 E_{n,1} + \pi_2 E_{n,2} + \cdots + \pi_{n+1} E_{n,n+1} &= 0 \\
\pi_1 + \pi_2 \quad \cdots + \pi_{n+1} &= 0
\end{aligned}
$$

is solved providing a solution for π_j $(j=1, \ldots, n+1)$. These values π_j are used to obtain the values of the new guess:

$$x_1 = E_{1,1}\ \pi_1 + E_{1,2}\ \pi_2 + \cdots + E_{1,n+1}\ \pi_{n+1}$$

$$x_2 = E_{2,1}\ \pi_1 + E_{2,2}\ \pi_2 + \cdots + E_{2,n+1}\ \pi_{n+1}$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$x_n = E_{n,1}\ \pi_1 + E_{n,2}\ \pi_2 + \cdots + E_{n,n+1}\ \pi_{n+1}$$

These equations are essentially the ones given in Reference 1.

The process of eliminating the "worst" of the previous points may be done either by eliminating the one giving the largest sum of squares of errors, as suggested in Reference 2, or using the weighted average method given in Formulae 3.

Acknowledgement

The cooperation of Mr. L. D. Nelson in illustrating the method and reviewing this memorandum is hereby gratefully acknowledged.

J. J. Schoch

1021-JJS-jld J. J. Schoch

Attachments
References, Figures 1-4

Copy to
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REFERENCES

1. The Secant Method for Simultaneous Nonlinear Equations, Comm. A. C. M. 2 (12), 1959, P. Wolfe.
2. "Trajectory and Targeting Computer Program", J. J. Schoch, Bellcomm Technical Memorandum TM-66-1021-2, March 1, 1966.
3. "Subroutine Wolfe", W. B. Gragg and J. M. Ortega, available at the Bellcomm Computer Library, August, 1964.
4. Solid Analytic Geometry, John M. H. Olmsted, Appleton Century Co., 1947.

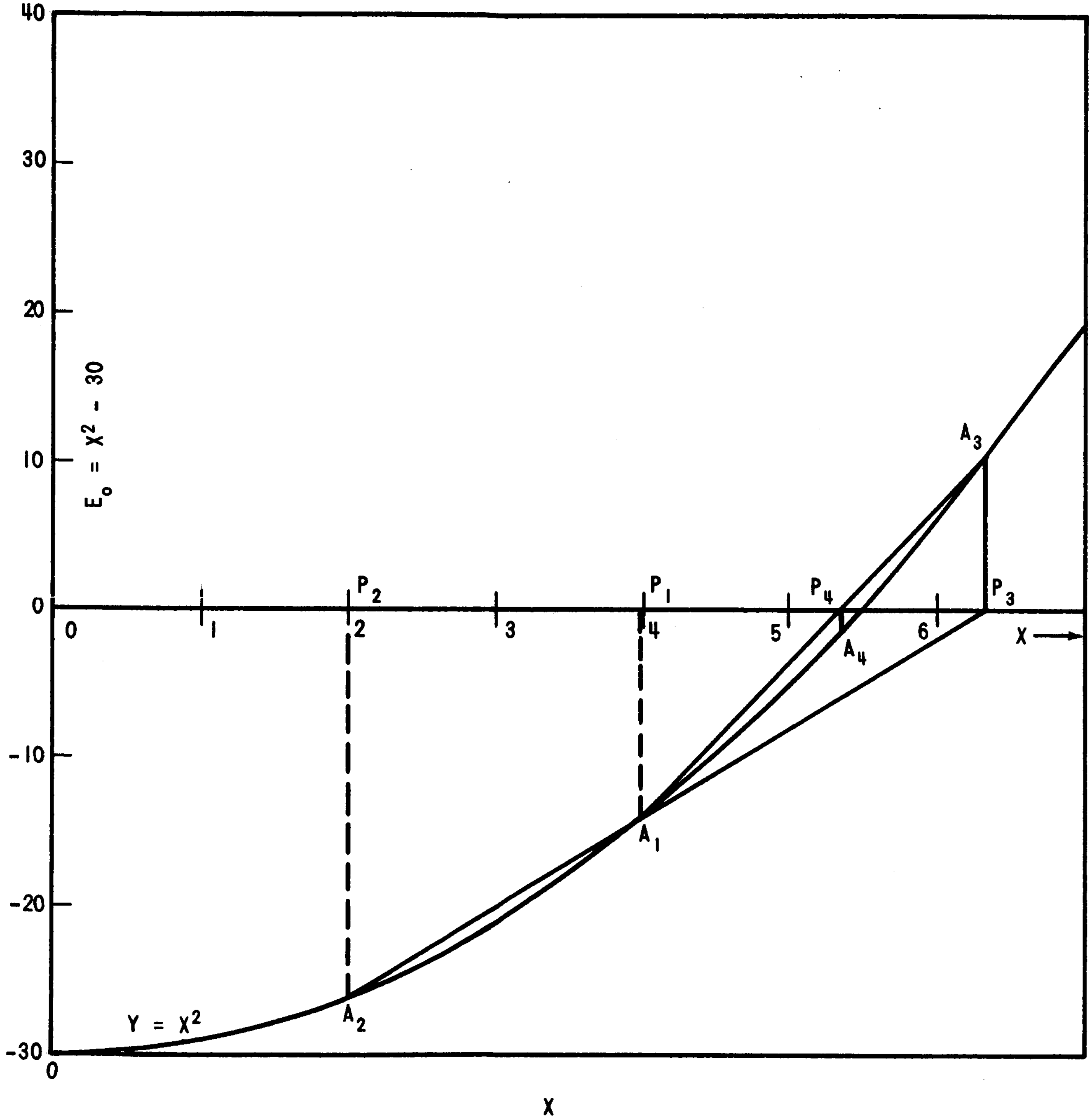
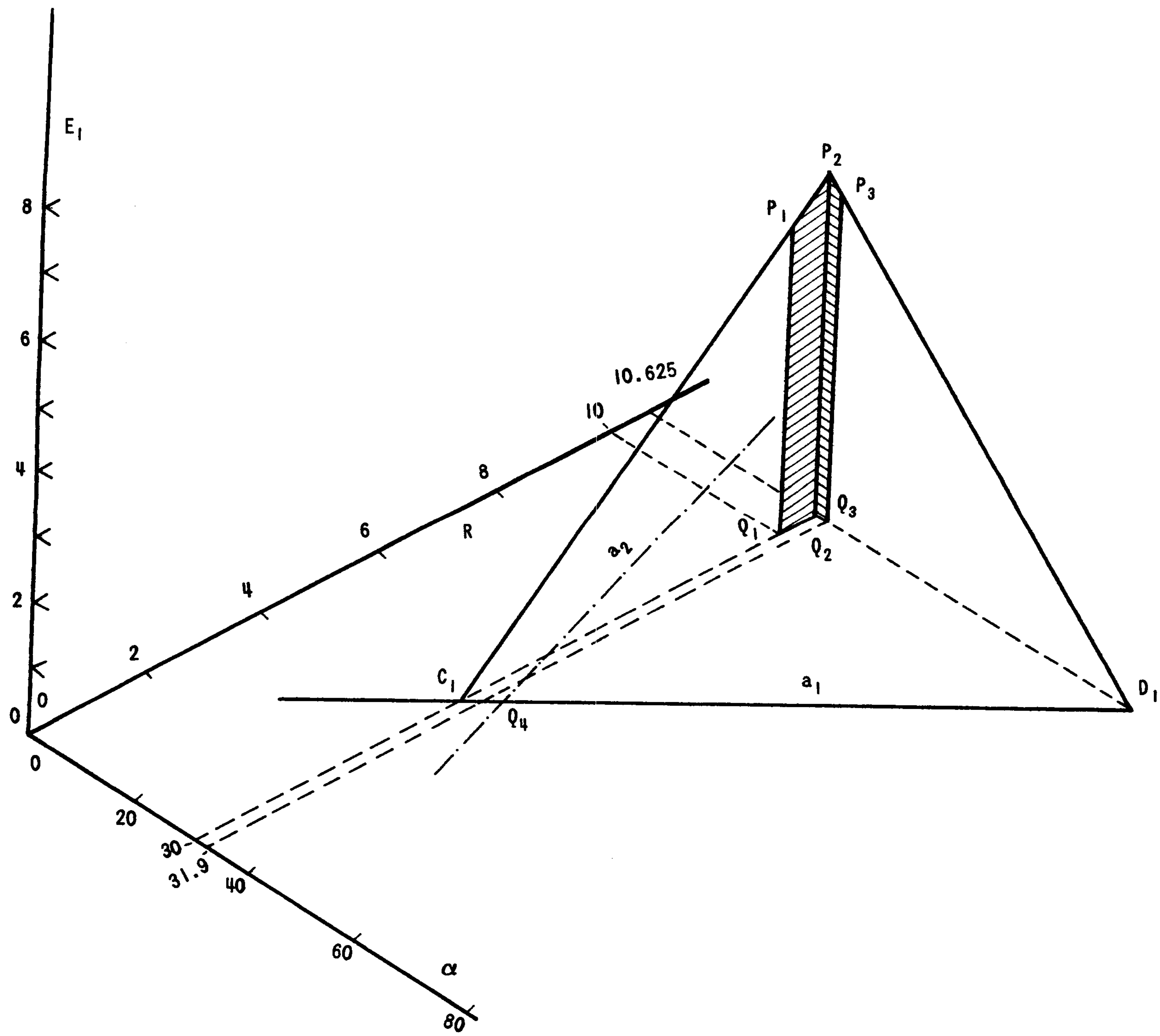

40
30
20
10
0
-10
-20
-30
$E_o = X^2 - 30$
0
1
2
3
4
5
6
$X \rightarrow$
P_2
P_1
P_4
P_3
A_3
A_4
A_1
A_2
$Y = X^2$
0
X

FIGURE I



E1
8
6
4
2
0
0
0
2
4
6
8
R
10
10.625
P1
P2
P3
Q1
Q2
Q3
Q4
C1
D1
a1
a2
20
30
31.9
40
60
80
α

FIGURE 2

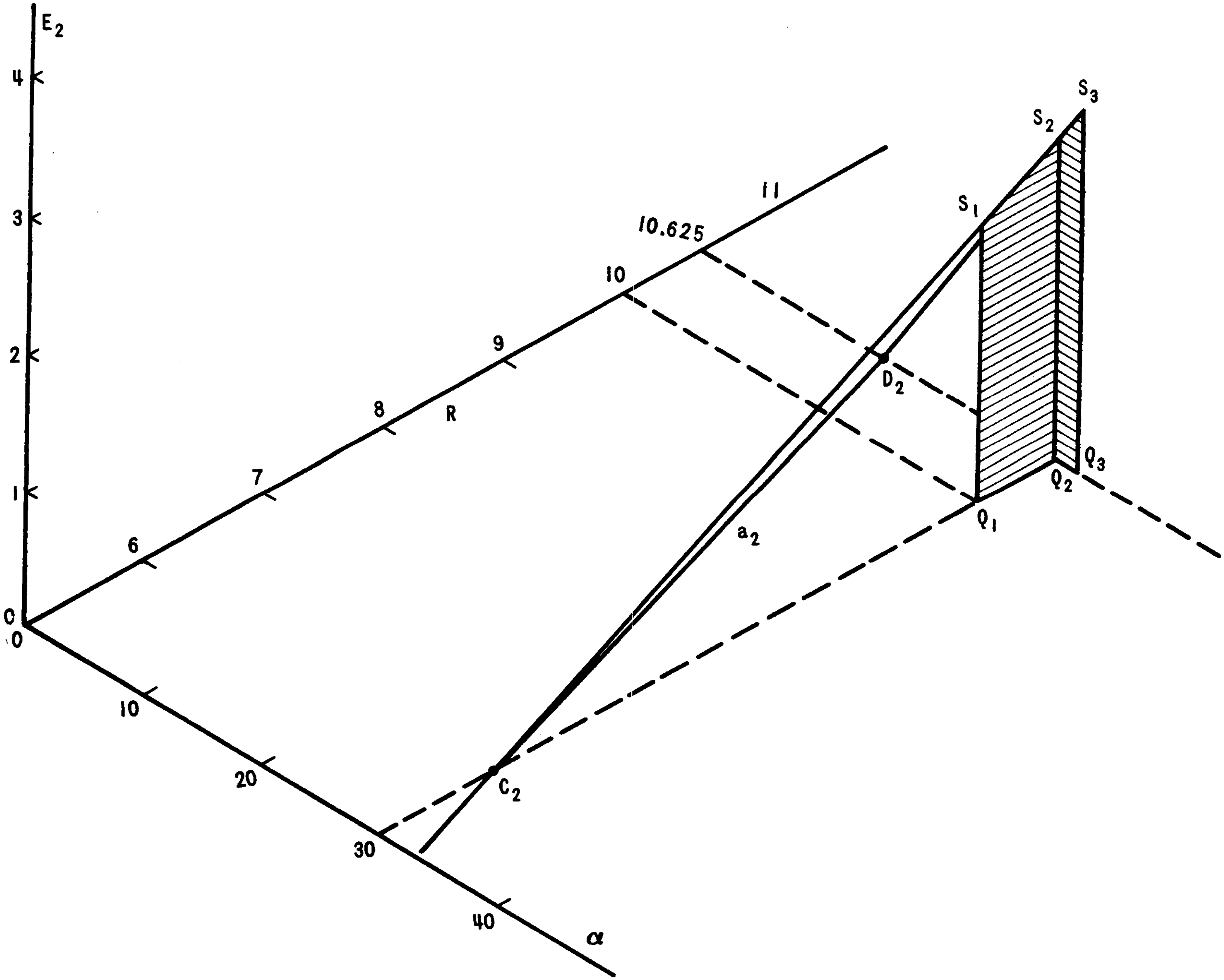


FIGURE 3

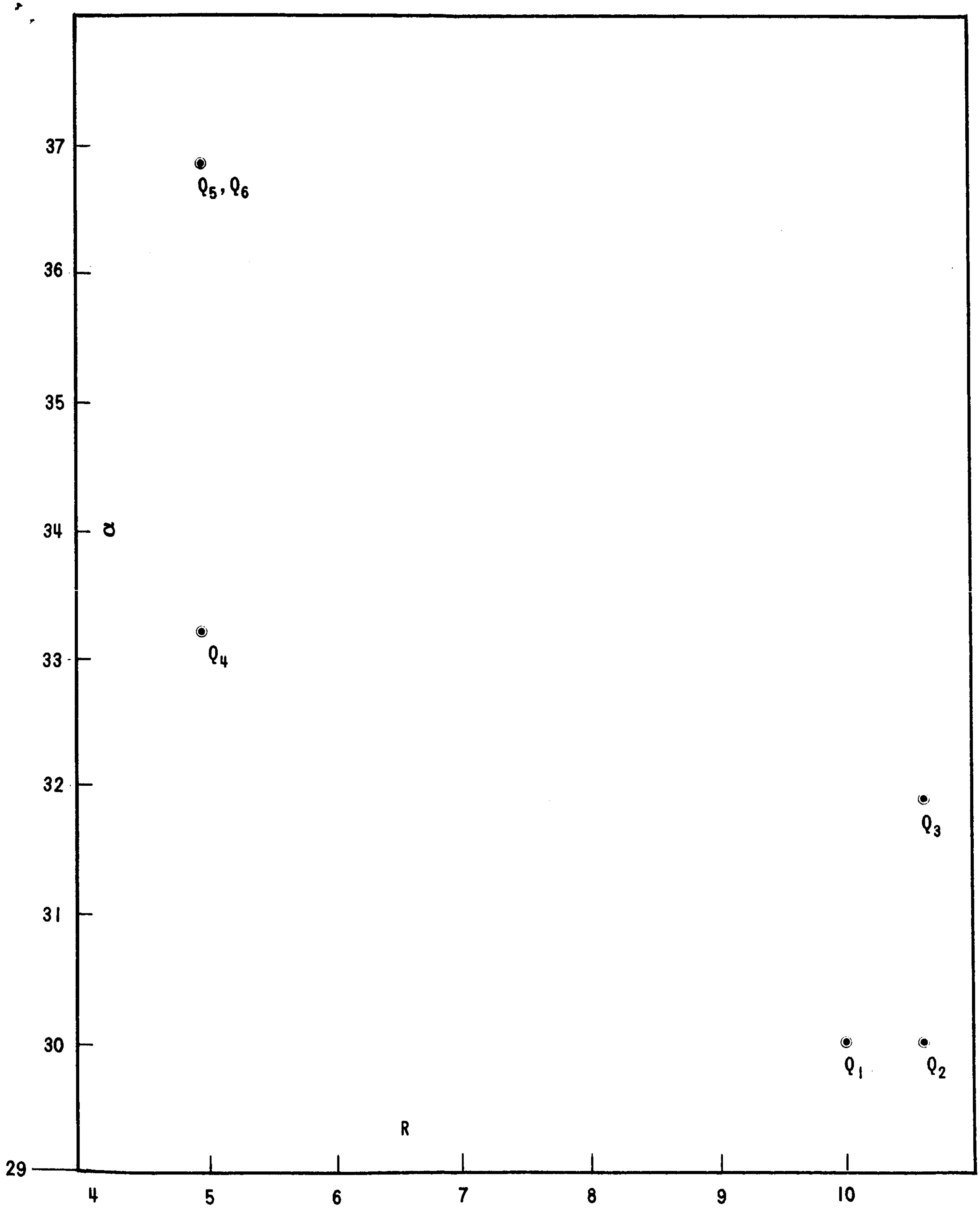


FIGURE 4